ON OSCILLATIONS OF A PHYSICAL PENDULUM HAVING CAVITIES FILLED WITH A VISCOUS LIQUID

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In [1] an asymptotic method was suggested for the investigation of non-stationary motions of a viscous incompressible liquid at large Reynolds numbers N_{Re}, arising as a result of the oscillation of various rigid bodies, either containing liquid or immersed in it, as well as at oscillations of liquid volumes having a free surface.

In the present paper the idea of this method is used for the investigation of small oscillations of a physical pendulum having cavities completely filled with an incompressible viscous liquid.

1. The motion of a liquid contained in the cavity of an oscillating pendulum (Fig. 1) is described by the equations of Navier-Stokes and by the equation of continuity

$$\frac{\partial \mathbf{V}'}{\partial t'} + (\mathbf{V}' \cdot \nabla')\,\mathbf{V}' = \nabla'\varphi' - \nu\,(\nabla' \times \boldsymbol{\Omega}')$$
$$\operatorname{div} \mathbf{V}' = 0 \tag{1.1}$$
$$\left(\boldsymbol{\Omega}' = \nabla' \times \mathbf{V}',\ \varphi' = -\frac{p}{\rho} - U\right)$$

Here U is the potential of mass forces acting on the liquid.

On the boundary of the cavity the condition of adhesion of the particles of the liquid to the walls of the cavity must be fulfilled, which gives the following boundary conditions:

$$u' = -y'\dot{\theta}, \quad v' = x'\dot{\theta}, \quad w' = 0 \quad \left(\dot{\theta} = \frac{d\theta}{dt'}\right) \tag{1.2}$$

Here θ is the angle of deviation of the pendulum from its equilibrium position, u', v' and w' are the components of the velocity vector $\mathbf{V}'$.

Let

$$\mathbf{V}' = \mathbf{V}'' + \mathbf{V}_0',$$

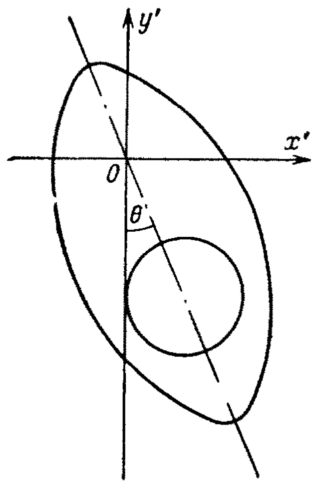

Fig. 1.

where $\mathbf{V}_0'$ is the velocity vector of the center of the cavity masses. Let us pass to a new system of coordinates (x'', y'', z'') with the origin at the center of the cavity masses and the axes parallel to the axes of the fixed system of coordinates (x', y', z').

In the new system of coordinates we shall have

$$\frac{\partial \mathbf{V}''}{\partial t'} + (\mathbf{V}'' \cdot \nabla'')\, \mathbf{V}'' = \nabla''\varphi'' - \nu\, (\nabla'' \times \mathbf{\Omega}''),$$

$$\operatorname{div} \dot{\mathbf{V}}'' = 0 \qquad (1.3)$$

Here

$$\mathbf{\Omega}'' = \nabla'' \times \mathbf{V}'', \qquad \varphi'' = -p/\rho - U - (\dot{\mathbf{V}}_0' \cdot \mathbf{r}')$$

On the boundary of the cavity

$$u'' = -y''\dot{\theta}, \qquad v'' = x''\dot{\theta}, \qquad w'' = 0 \qquad (1.4)$$

Let us refer all the values to the characteristic scales. Let

$$t' = Tt, \quad x'' = Rx, \quad y'' = Ry, \quad z'' = Rz, \quad \mathbf{V}'' = \frac{\alpha}{T} R\mathbf{V}$$
$$\mathbf{\Omega}'' = \frac{\alpha}{T}\mathbf{\Omega}, \quad \varphi'' = \alpha\frac{R^2}{T^2}\varphi, \quad \theta = \alpha\vartheta, \quad N_{\mathrm{Re}} = \frac{R^2}{\nu T} \qquad (1.5)$$

where T is the characteristic time of oscillation, R is the characteristic dimension of the cavity, α is the characteristic amplitude and N_{Re} is the Reynolds number.

In dimensionless variables the equations (1.3) and the boundary conditions (1.4) become

$$\frac{\partial \mathbf{V}}{\partial t} + \alpha\, (\mathbf{V} \cdot \nabla)\, \mathbf{V} = \nabla\varphi - \frac{1}{N_{\mathrm{Re}}}(\nabla \times \mathbf{\Omega}), \qquad \operatorname{div} \mathbf{V} = 0 \qquad (1.6)$$

On the boundary of the cavity

$$u = -y\dot{\vartheta}, \qquad v = x\dot{\vartheta}, \qquad w = 0 \tag{1.7}$$

In the system of equations (1.6) the nonlinear terms have the characteristic amplitude α as factor.

We shall consider oscillations with a small amplitude, and in what follows we shall linearize the equations (1.6) neglecting magnitudes of order α.

The linearized system of equations is

$$\frac{\partial \mathbf{V}}{\partial t} = \nabla\varphi - \frac{1}{N_{\mathrm{Re}}}(\nabla \times \mathbf{\Omega}), \qquad \operatorname{div}\mathbf{V} = 0 \tag{1.8}$$

The purpose of the present paper is to investigate such forms of oscillations for which the solution may be represented in the form

$$\vartheta(t) = ce^{\sigma t}, \qquad \mathbf{V} = ce^{\sigma t}\mathbf{U}(x, y, z) \tag{1.9}$$

Putting $\varphi = ce^{\sigma t}\Phi(x, y, z), \mathbf{\Omega} = ce^{\sigma t}\mathbf{\Psi}(x, y, z)$, we obtain for such motions

$$\sigma\mathbf{U} = \nabla\Phi - \frac{1}{N_{\mathrm{Re}}}(\nabla \times \mathbf{\Psi}), \qquad \operatorname{div}\mathbf{U} = 0 \tag{1.10}$$

On the boundary of the cavity

$$U_x = -y\sigma, \qquad U_y = x\sigma, \qquad U_z = 0 \tag{1.11}$$

The relation (1.10) shows that the vector $\mathbf{U}$ represents the sum of the potential and the solenoidal vectors. Such a representation permits separation of the equations, obtaining for each unknown function a separate equation.

Indeed, taking on each side of the first equation of the system (1.10), the operation div, we obtain

$$\triangle\Phi = 0 \tag{1.12}$$

i.e. the function Φ is harmonic in the volume occupied by the liquid. Taking on both sides of the same equation the operation rot, we obtain

$$\sigma\mathbf{\Psi} = \frac{1}{N_{\mathrm{Re}}}\triangle\mathbf{\Psi} \tag{1.13}$$

Thus, each of the unknown functions Φ, Ψ satisfy a separate equation.

On the boundary of the cavity these functions are connected by the following boundary conditions:

$$\frac{\partial\Phi}{\partial x}-\frac{1}{N_{\mathrm{Re}}}\left(\frac{\partial\Psi_z}{\partial y}-\frac{\partial\Psi_y}{\partial z}\right)=-y\sigma^2$$

$$\frac{\partial\Phi}{\partial y}-\frac{1}{N_{\mathrm{Re}}}\left(\frac{\partial\Psi_x}{\partial z}-\frac{\partial\Psi_z}{\partial x}\right)=x\sigma^2$$

$$\frac{\partial\Phi}{\partial z}-\frac{1}{N_{\mathrm{Re}}}\left(\frac{\partial\Psi_y}{\partial x}-\frac{\partial\Psi_x}{\partial y}\right)=0$$

Here Ψ_x, Ψ_y, Ψ_z are the components of the vector Ψ.

The fourth boundary condition determining on the boundary the projection of the vector Ψ onto the normal to the cavity surface will be given in the following in a specially selected curvilinear system of coordinates.

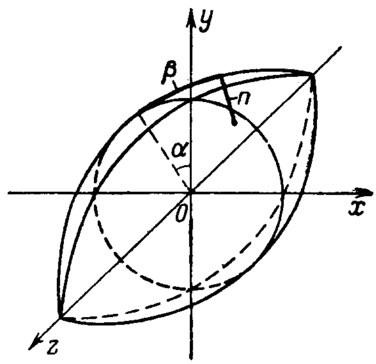

Fig. 2.

Let us consider cavities representing rotation figures whose axes are perpendicular to the oscillation plane of the body.

Let us introduce a curvilinear system of coordinates connected with the surface of the cavity. In the case under consideration it is convenient to introduce the following coordinates: n is the distance along the inner normal to the cavity surface measured from the cavity surface towards the interior of the cavity, α is the angle determining the position of the meridian plane, β is the length of the arc along the meridian (Fig. 2). The variables in the system of coordinates (x, y, z) are connected with the variables n, α, β in the curvilinear system of coordinates by the following relations:

$$x=-\left[r_0+\int_0^\beta \sin\gamma(\beta)\,d\beta-n\cos\gamma(\beta)\right]\sin\alpha=-[r(\beta)-n\cos\gamma(\beta)]\sin\alpha$$

$$y=\left[r_0+\int_0^\beta \sin\gamma(\beta)\,d\beta-n\cos\gamma(\beta)\right]\cos\alpha=[r(\beta)-n\cos\gamma(\beta)]\cos\alpha$$

$$z=\int_0^\beta \cos\gamma(\beta)\,d\beta+n\sin\gamma(\beta)$$

Here r_0 is the radius of the circle obtained by intersecting the cavity with a plane perpendicular to the axis of symmetry of the cavity and passing through the center of the cavity masses; $r(\beta)$ is the radius of the circumference formed by the points with same coordinate β on the surface of the cavity; $\gamma(\beta)$ is the angle between the tangent to the meridian at the point with the coordinate β and the positive direction of the z-axis. Let us write in the new coordinate system the equations (1.13) and the boundary conditions. To abbreviate the writing, set

$$\Psi_x = \Psi_1, \qquad \Psi_y = \Psi_2, \qquad \Psi_z = \Psi_3$$

Then we have in the curvilinear system of coordinates

$$\sigma\Psi_i = \frac{1}{N_{\mathrm{Re}}\,[r(\beta) - n\cos\gamma(\beta)]\,[1 + n\gamma'(\beta)]}\left[\frac{\partial}{\partial\alpha}\left(\frac{1 + n\gamma'(\beta)}{r(\beta) - n\cos\gamma(\beta)}\,\frac{\partial\Psi_i}{\partial\alpha}\right) + \right.$$

$$\left. + \frac{\partial}{\partial\beta}\left(\frac{r(\beta) - n\cos\gamma(\beta)}{1 + n\gamma'(\beta)}\,\frac{\partial\Psi_i}{\partial\beta}\right) + \frac{\partial}{\partial n}\left([r(\beta) - n\cos\gamma(\beta)]\,[1 + n\gamma'(\beta)]\,\frac{\partial\Psi_i}{\partial n}\right)\right]$$

$$\left(\gamma'(\beta) = \frac{d\gamma(\beta)}{d\beta}\right) \qquad (i = 1, 2, 3) \tag{1.14}$$

On the cavity boundary, for $n = 0$

$$\frac{\partial\Phi}{\partial n} - \frac{1}{N_{\mathrm{Re}}\,r(\beta)}\left[\frac{\partial\Psi_\beta}{\partial\alpha} - \frac{\partial r(\beta)\,\Psi_\alpha}{\partial\beta}\right] = 0 \tag{1.15}$$

$$\frac{1}{r(\beta)}\,\frac{\partial\Phi}{\partial\alpha} - \frac{1}{N_{\mathrm{Re}}}\left[\frac{\partial\Psi_n}{\partial\beta} - \frac{\partial\,(1 + n\gamma'(\beta))\,\Psi_\beta}{\partial n}\right] = r(\beta)\,\sigma^2 \tag{1.16}$$

$$\frac{\partial\Phi}{\partial\beta} - \frac{1}{N_{\mathrm{Re}}\,r(\beta)}\left[\frac{\partial\,(r(\beta) - n\cos\gamma(\beta))\,\Psi_\alpha}{\partial n} - \frac{\partial\Psi_n}{\partial\alpha}\right] = 0 \tag{1.17}$$

We give the fourth boundary condition. The normal component Ψ_n of the vortex vector $\boldsymbol{\Psi}$ is determined by the tangential components U_α and U_β of the velocity vector $\mathbf{U}$ in the following way:

$$\Psi_n = \frac{1}{[r(\beta) - n\cos\gamma(\beta)]\,[1 + n\gamma'(\beta)]}\left[\frac{\partial\,(1 + n\gamma'(\beta))\,U_\beta}{\partial\alpha} - \frac{\partial\,(r(\beta) - n\cos\gamma(\beta))\,U_\alpha}{\partial\beta}\right]$$

On the cavity boundary the vector $\mathbf{U}$ is known, consequently its components U_α and U_β for $n = 0$ are known. The differentiation with respect to α and β for n = const is possible, therefore it is not difficult to calculate Ψ_n for $n = 0$. After simple computations we obtain

$$\Psi_n = -\,2\sin\gamma(\beta)\,\sigma \tag{1.18}$$

This relation concludes the system of the boundary conditions for the unknown functions Φ, Ψ_i.

Ψ_α, Ψ_β, Ψ_n in the relations (1.15) to (1.18) represent the components of the vector $\boldsymbol{\Psi}$ in the curvilinear coordinate system. They are connected with the components Ψ_i by the relations

$$\begin{aligned}
\Psi_\alpha &= -\Psi_1 \cos\alpha - \Psi_2 \sin\alpha \\
\Psi_\beta &= -\Psi_1 \sin\gamma(\beta)\sin\alpha + \Psi_2 \sin\gamma(\beta)\cos\alpha + \Psi_3\cos\gamma(\beta) \qquad (1.19) \\
\Psi_n &= \Psi_1 \cos\gamma(\beta)\sin\alpha - \Psi_2\cos\gamma(\beta)\cos\alpha + \Psi_3\sin\gamma(\beta)
\end{aligned}$$

Let us assume that the parameters of the pendulum ensure a sufficiently large Reynolds number. Set

$$\frac{1}{N_{\mathrm{Re}}} = \varepsilon^2 \qquad (1.20)$$

where ε is a dimensionless small parameter.

The idea of constructing the solution for large Reynolds numbers given in [1] is analogous to the idea of constructing the boundary layer. It is assumed that the vortices existing in the oscillating liquid contained in the cavity of the pendulum are essentially concentrated in a thin layer at the walls of the cavity. This in turn permits one to assume that the derivatives of the components of the vector Ψ along the normal to the cavity surface are essentially larger than along the tangential directions. Let us introduce a "stretching" of the independent variable n. Set

$$n = \varepsilon\eta$$

We seek the solution of the posed problem in the form of series of powers of a small parameter ε

$$\Phi = \Phi_0 + \varepsilon\Phi_1 + \ldots, \qquad \Psi_i = \frac{1}{\varepsilon}\Psi_{0i} + \Psi_{1i} + \ldots \qquad (i = 1, 2, 3) \qquad (1.21)$$

Substituting the series (1.21) into the equations and boundary conditions, and equating to zero the sum of the coefficients of the parameter ε with zero exponent, we obtain the following problem for determining the functions Φ_0 and Ψ_{0i}

$$\Delta\Phi_0 = 0, \qquad \sigma\Psi_{0i} = \frac{\partial^2\Psi_{0i}}{\partial\eta^2} \qquad (i = 1, 2, 3) \qquad (1.22)$$

On the cavity boundary, for $n = 0$

$$\frac{\partial\Phi_0}{\partial n} = 0, \qquad \frac{1}{r(\beta)}\frac{\partial\Phi_0}{\partial\alpha} + \frac{\partial\Psi_{0\beta}}{\partial\eta} = r(\beta)\sigma^2, \qquad \frac{\partial\Phi_0}{\partial\beta} - \frac{\partial\Psi_{0\alpha}}{\partial\eta} = 0, \qquad \Psi_{0n} = 0 \qquad (1.23)$$

From the first relation of the conditions (1.23) we see that the normal derivative of the harmonic function Φ_0 on the cavity boundary is equal to zero. Therefore, without loss of generality, one may assume $\Phi_0 \equiv \text{const}$.

The components of the vector Ψ in a curvilinear system of coordinates are related to the components of this vector in a Cartesian coordinate system by expressions (1.19). These relations are linear with respect to Ψ_{01}, Ψ_{02} and Ψ_{03} and the coefficients of Ψ_{01}, Ψ_{02} and Ψ_{03} do not depend on η; consequently the functions $\Psi_{0\alpha}$, $\Psi_{0\beta}$ and Ψ_{0n} satisfy the same equations as Ψ_{0i}. Thus

$$\sigma\Psi_{0\alpha} = \frac{\partial^2\Psi_{0\alpha}}{\partial\eta^2}, \qquad \sigma\Psi_{0\beta} = \frac{\partial^2\Psi_{0\beta}}{\partial\eta^2}, \qquad \sigma\Psi_{0n} = \frac{\partial^2\Psi_{0n}}{\partial\eta^2} \tag{1.24}$$

On the cavity boundary these functions satisfy the following boundary conditions

$$\frac{\partial\Psi_{0\beta}}{\partial\eta} = r(\beta)\,\sigma^2, \qquad \frac{\partial\Psi_{0\alpha}}{\partial\eta} = 0, \qquad \Psi_{0n} = 0 \tag{1.25}$$

The general solution of the equation satisfied by $\Psi_{0\alpha}$, $\Psi_{0\beta}$ and Ψ_{0n} has the form

$$u = c_1 e^{\sqrt{\sigma}\eta} + c_2 e^{-\sqrt{\sigma}\eta}$$

Let $\operatorname{Re}\sqrt{\sigma} > 0$. By assumption, far away from the cavity walls the vortices are absent. As in the theory of the boundary layer, we shall assume the value $\eta = \infty$ corresponding to the inner points of the cavity sufficiently distant from the boundary. Then, according to the assumption of absence of vortices for $\eta = \infty$ we shall have $c_1 = 0$.

The arbitrary constant of integration c_2 is determined from the conditions (1.25). Finally we obtain for the functions $\Psi_{0\alpha}$, $\Psi_{0\beta}$, Ψ_{0n}

$$\Psi_{0\alpha} = 0, \qquad \Psi_{0\beta} = -\,r(\beta)\,\sigma\sqrt{\sigma}\,e^{-\sqrt{\sigma}\eta}, \qquad \Psi_{0n} = 0 \tag{1.26}$$

We limit ourselves to determining only the first terms of the series (1.21). Within this accuracy let us write the components of the vector of the absolute velocity $\mathbf{V}'$ in the rigid coordinate system with the origin at the point of suspension of the pendulum

$$\begin{aligned} u' &= c\frac{R}{T}\,[l\sigma - r(\beta)\,\sigma\cos\alpha\, e^{-\sqrt{\sigma}\eta} + O(\varepsilon)]\,e^{\sigma t}, \\ v' &= -c\frac{R}{T}\,[r(\beta)\,\sigma\sin\alpha\, e^{-\sqrt{\sigma}\eta} + O(\varepsilon)]\,e^{\sigma t}, \end{aligned} \qquad w' = 0 \tag{1.27}$$

where l is the distance of the cavity masses from the axis of suspension referred to the characteristic dimension of the cavity. The obtained solution (1.27) is asymptotic. For it holds the following result whose proof is not given here. The modulus of the difference

$$|\mathbf{V}' - \mathbf{V}_0'| \leqslant O(\varepsilon)$$

Here $\mathbf{V}'$ is the exact solution of the linearized problem, and $\mathbf{V}_0'$ is the obtained approximate solution of the same problem.

Thus, if the Reynolds number is sufficiently large the approximate solution ensures a good accuracy.

2. In order to solve completely the problem posed, it is necessary to determine the as yet unknown constant σ. To this end it is necessary to use the equation for the oscillations of a pendulum which may be obtained employing the theorem on the variation of the moment of momentum of the system. We have

$$\frac{d}{dt}\int_{D_1} (\mathbf{r}' \times \rho_1\mathbf{V}_1')\, d\tau + \frac{d}{dt}\int_{D} (\mathbf{r}' \times \rho\mathbf{V}')\, d\tau = \int_{D_1} (\mathbf{r}' \times \rho_1\mathbf{g})\, d\tau + \int_{D} (\mathbf{r}' \times \rho\mathbf{g})\, d\tau \tag{2.1}$$

Here D_1 is the volume of the solid body; D is the volume of the cavity; ρ_1 and $\mathbf{V}_1'$ are the density and the velocity of particles of the body, respectively; ρ and $\mathbf{V}'$ are the density and the velocity of the points of the fluid, respectively; $\mathbf{g}$ is the vector of acceleration of the force of gravity; $\mathbf{r}'$ is the radius vector from the axis of rotation to the point of the body or of the liquid. In our case, the equation of the moments (2.1) gives a projection which only on the z'-axis is not identically equal to zero.

The integrals in (2.1) (returning to dimensional variables according to (1.5)) may be brought to the following form

$$\left\{\frac{d}{dt}\int_{D_1} (\mathbf{r}' \times \rho_1\mathbf{V}_1')\, d\tau\right\}_{Z'} = cM_1k^2\sigma'^2e^{\sigma't'} \qquad \left(\sigma' = \frac{\sigma}{T}\right) \tag{2.2}$$

$$\left\{\frac{d}{dt}\int_{D} (\mathbf{r}' \times \rho\mathbf{V}')\, d\tau\right\}_{Z'} = c\,(Ml'^2\sigma'^2 + \sqrt{\nu}\, MQ\sigma'^{\frac{3}{2}})\, e^{\sigma't'} \qquad \left(MQ = 2\pi\rho\int_{\beta_1'}^{\beta_2'} r'^3(\beta')\, d\beta'\right) \tag{2.3}$$

$$\left\{\int_{D_1} (\mathbf{r}' \times \rho_1\mathbf{g})\, d\tau + \int_{D} (\mathbf{r}' \times \rho\mathbf{g})\, d\tau\right\}_{Z'} = -\,cg\,(M_1l_1' + Ml')\, e^{\sigma't'} \tag{2.4}$$

Here M_1 is the mass of the solid body; M is the mass of the fluid; k is the radius of inertia of the solid body with respect to the suspension axis of the pendulum; l' is the distance from the suspension axis to the center of the cavity masses; l_1' is the distance from the suspension axis to the center of the masses of the solid body; β_1', β_2' are the coordinates of the poles of the cavity.

Substituting (2.2), (2.3) and (2.4) into (2.1) we obtain the following equation for determining σ':

$$\sigma'^2 + \lambda\sqrt{\nu}\,\sigma'^{\frac{3}{2}} + \omega^2 = 0 \quad \left(\lambda = \frac{MQ}{M_1k^2 + Ml'^2},\ \omega^2 = g\frac{M_1l_1' + Ml'}{M_1k^2 + Ml'^2}\right) \tag{2.5}$$

Put $\sigma' = \nu\lambda^2 x$. Then (2.5) becomes

$$x^2 + x^{\frac{3}{2}} + \frac{\omega^2}{\nu^2\lambda^4} = 0 \quad \text{or} \quad y^4 + y^3 + q^2 = 0 \qquad \left(\sqrt{x} = y,\ \frac{\omega^2}{\nu^2\lambda^4} = q^2\right) \tag{2.6}$$

Let us prove that this equation has only two roots satisfying the condition $\operatorname{Re} y > 0$. Put $y = \alpha + i\beta$.

Substituting into (2.6) we obtain for determining α and β the system

$$\begin{gathered}(\alpha^2 - \beta^2)^2 - 4\alpha^2\beta^2 + \alpha^3 - 3\alpha\beta^2 + q^2 = 0\\ 4\alpha^3 - 4\alpha\beta^2 - \beta^2 + 3\alpha^2 = 0\end{gathered} \tag{2.7}$$

Determining β^2 from the second equation of the system (2.7) and substituting it into the first equation we obtain

$$64\alpha^6 + 96\alpha^5 + 48\alpha^4 + 8\alpha^3 - q^2(16\alpha^2 + 8\alpha + 1) = 0, \quad \beta_{1,2} = \pm\alpha\sqrt{\frac{4\alpha+3}{4\alpha+1}} \tag{2.8}$$

There is only one change of sign of the coefficients in the first equation (2.8). By the Cartesian rule this polynomial has one positive root. We are not interested in the negative roots, since for them the condition $\operatorname{Re}\sqrt{x}$ is violated. From the second relation of the system (2.8) two values of β are determined differing only in sign. Thus, the equation (2.6) has only two roots for the condition $\operatorname{Re} y > 0$.

Let us seek the solution of equation (2.6) in two limiting cases, when q is large and when q is small.

We recall that the solution of the problem on the motion of the fluid in the cavity of the pendulum was obtained for a large Reynolds number. The magnitude of the Reynolds number in turn depends essentially on the characteristic time of one oscillation $\pi/|\operatorname{Im}\sigma'|$, which will be determined only after having solved the equation (2.6). Consequently, from all the possible solutions of the equation (2.6) corresponding to the various parameters of the pendulum, one can use only those which ensure a sufficiently large Reynolds number. In the following we shall see that in the limiting cases under consideration of large and small q it is always possible to select such pendulum parameters that the Reynolds number remains large.

Let us consider the case of large q. We seek the solution of equation (2.6) in form of a series of powers of $1/\sqrt{q}$

$$y = \sqrt{q}\left(y_0 + \frac{1}{\sqrt{q}}y_1 + \ldots\right) \tag{2.9}$$

Substituting (2.9) into (2.6) and equating to zero the sum of the coefficients of equal powers of $1/\sqrt{q}$, we obtain equations for determining the unknowns y_0, y_1 etc.

For y_0 we have

$$y_0^4 + 1 = 0, \qquad \operatorname{Re} y_0 > 0$$

From where we find

$$y_{01} = \frac{\sqrt{2}}{2}(1+i), \qquad y_{02} = \frac{\sqrt{2}}{2}(1-i)$$

To determine y_1 we have

$$4y_1 + 1 = 0 \quad \text{or} \quad y_1 = -{}^1/_4$$

With an accuracy of up to the order of $1/\sqrt{q}$ inclusive, we find

$$y = \frac{\sqrt{2q}}{2} - \frac{1}{4} \pm i\frac{\sqrt{2q}}{2}$$

or

$$\sigma' = -\sqrt{\nu}\frac{\lambda\sqrt{\omega}}{2\sqrt{2}} \pm i\left(\omega - \sqrt{\nu}\frac{\lambda\sqrt{\omega}}{2\sqrt{2}}\right)$$

$$(\sigma' = \nu\lambda^2 y^2) \tag{2.10}$$

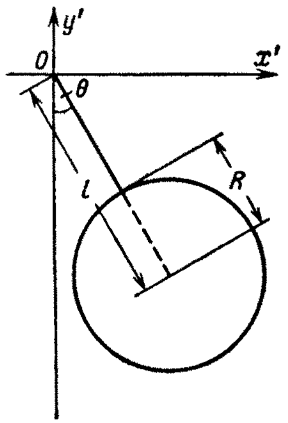

Fig. 3.

Thus, in the case considered the frequency η and the amplitude A of the oscillations of the pendulum will be

$$n = \omega - \sqrt{\nu}\frac{\lambda\sqrt{\omega}}{2\sqrt{2}}, \qquad A = \exp\left(-\sqrt{\nu}\frac{\lambda\sqrt{\omega}}{2\sqrt{2}}t'\right)$$

respectively.

For $\nu \to 0$ the damping coefficient tends toward zero, and the frequency of oscillations tends toward ω corresponding to oscillations of a pendulum with an ideal liquid. Let us compute by what fraction the amplitude decreases during $T = \pi/|\operatorname{Im}\sigma'|$. Denote this fraction by Δ. We have

$$\Delta = 1 - \frac{\exp \operatorname{Re}(\sigma' t_2')}{\exp \operatorname{Re}(\sigma' t_1')} = 1 - \exp\frac{\pi \operatorname{Re}\sigma'}{|\operatorname{Im}\sigma'|} \tag{2.11}$$

Determining $\mathrm{Re}\,\sigma'$ and $|\mathrm{Im}\,\sigma'|$ from (2.10) and substituting into (2.11) we obtain

$$\Delta = 1 - \exp\left(-\frac{\pi}{2\sqrt{2q}-1}\right) \approx \frac{\pi}{2\sqrt{2q}} \qquad (\Delta \to 0 \text{ for } q \to \infty)$$

It is easy to see that the considered case of large q corresponds to oscillations close to the oscillations of a pendulum with an ideal liquid.

Let us consider the simplest example: let the pendulum represent a weightless spherical envelope filled with a viscous liquid and connected with the axis of suspension (Fig. 3) by means of a weightless rod. For such a pendulum we have

$$\lambda = \frac{2R}{l^2}, \qquad \omega^2 = \frac{g}{l}, \qquad q = \frac{\sqrt{gl^3}}{4\nu}\frac{l^2}{R^2}$$
$$\sigma' = -\sqrt{\nu}\,\frac{R\sqrt[4]{g}}{l^2\sqrt[4]{4l}} \pm i\left(\sqrt{\frac{g}{l}} - \sqrt{\nu}\,\frac{R\sqrt[4]{g}}{l^2\sqrt[4]{4l}}\right) \tag{2.12}$$

The characteristic time of one oscillation with an accuracy up to magnitudes of the order $1/\sqrt{q}$ is determined by

$$T \approx \pi\sqrt{l/g} \tag{2.13}$$

With the increase of the length of the pendulum l the magnitude T increases. In this case there appears the danger of an inadmissible decrease of the Reynolds number. Let us assume a value of the number $N_{\mathrm{Re}} \geqslant 10^4$ and let us estimate the admissible value of the length l of the pendulum for various values of the radius R. By definition $N_{\mathrm{Re}} = R^2/\nu T$.

To keep $N_{\mathrm{Re}} \geqslant 10^4$ it is necessary to require

$$T \leqslant \frac{R^2}{\nu 10^4} \quad \text{or} \quad \frac{l}{R} \leqslant \frac{R^3 g}{\nu^2\pi^2 10^8}$$

Let the pendulum be filled with water at $t = 20°\mathrm{C}$ ($\nu = 1.01 \times 10^{-6}$ m^2/sec). For $R = 0.1$ m we have $l/R \leqslant 10$, i.e. $l \leqslant 1$ m. For $R = 1$ m we obtain $l/R \leqslant 10^4$, i.e. $l \leqslant 10$ km. With an increase of R the admissible value of the ratio l/R increases proportionally to R^3. Thus, the range of the applicability of the solution (2.12) is sufficiently large.

Let us consider now the case of limiting small q values. We note that small q may be realized not only by increasing the kinematic coefficient of viscosity ν, which may lead to the diminution of the number N_{Re}. Indeed, on the example of the above considered pendulum it is seen that this may be obtained by diminishing the length l of the pendulum or by increasing the radius R of the cavity.

Let us represent (2.6) in the form

$$y^3 (y + 1) = - q^2 \tag{2.14}$$

It follows that

$$|y^3| \, |y + 1| = |- q^2| = q^2$$

With a decrease of q the product $|y^3| \, |y + 1|$ decreases. In this process the factor $|y + 1|$ cannot tend toward zero, since in the opposite case the condition $\operatorname{Re} y > 0$ is violated. Consequently, the modulus of the root sought has the order of smallness $q^{2/3}$. Proceeding from this we seek the solution of the equation (2.6) in form of the series

$$y = q^{2/3} (y_0 + q^{2/3} y_1 + \ldots)$$

For determining the unknowns y_0 and y_1 we obtain

$$y_0^3 + 1 = 0, \qquad y_0^2 + 3y_1 = 0$$

Wherefrom we find

$$y_{01} = \tfrac{1}{2} + i \tfrac{\sqrt{3}}{2}, \qquad y_{02} = \tfrac{1}{2} - i \tfrac{\sqrt{3}}{2}, \qquad y_{11} = \tfrac{1}{6} - i \tfrac{\sqrt{3}}{6}, \qquad y_{12} = \tfrac{1}{6} + i \tfrac{\sqrt{3}}{6}$$

With an accuracy up to magnitudes of the order $q^{2/3}$ inclusive we have

$$y = \tfrac{1}{2} q^{2/3} [(1 + \tfrac{1}{3} q^{2/3}) \pm i \sqrt{3} (1 - \tfrac{1}{3} q^{2/3})]$$

For σ' we obtain the expression

$$\sigma' = \frac{\omega^{4/3}}{2\nu^{1/3}\lambda^{2/3}} \left[- \left(1 - \frac{4}{3} \frac{\omega^{2/3}}{\nu^{2/3}\lambda^{4/3}} \right) \pm i \frac{\sqrt{3}}{2} \right] \tag{2.15}$$

Or limiting ourselves to magnitudes of the order $q^{4/3}$

$$\sigma' = \left(\frac{\omega^4}{\nu\lambda^2} \right)^{1/3} \left(- \frac{1}{2} \pm i \sqrt{3} \right)$$

As in the previous case, let us determine for which part the amplitude decreases from its initial value during the characteristic time T.

We have

$$\Delta = 1 - \exp \pi \operatorname{Re} \sigma' / |\operatorname{Im} \sigma'|$$

Using for determining $\operatorname{Re} \sigma'$ and $|\operatorname{Im} \sigma'|$ the expression (2.15), we obtain

$$\Delta = 1 - \exp\left[-\frac{\pi}{\sqrt{3}}\left(1 - \frac{4}{3} q^{2/3}\right)\right]$$

$$\left(\Delta \to 1 - \exp\left(-\frac{\pi}{\sqrt{3}}\right) \approx 0.84 \quad \text{as} \quad q \to 0\right) \tag{2.16}$$

It is evident that the considered case sharply differs from the case of oscillations of a pendulum with an ideal liquid. It turns out that it is possible to select the cavity of the pendulum so large or the length of the pendulum so small that it will not be possible to neglect the effect of viscosity on the motion of the pendulum in spite of the presence of large Reynolds numbers. In the case of limiting small q's during one swing the amplitude may diminish by a magnitude of the order of 84% of its initial value.

We show on the example of a pendulum represented on Fig. 3 that by increasing the radius R of the cavity at constant l and ν it is possible to achieve any desired large numbers N_{Re} and any desired small q. For the pendulum under consideration we have

$$\sigma' = \left(\frac{g^2 l^2}{4R_\nu^2}\right)^{1/3}\left(-\frac{1}{2} \pm i\,\frac{\sqrt{3}}{2}\right) \tag{2.17}$$

$$q = \frac{\sqrt{gl^3}}{4\nu}\,\frac{l^2}{R^2}, \qquad N_{\mathrm{Re}} = \frac{\sqrt{3}}{2\pi}\,\frac{R^{4/3}}{\nu}\left(g^2\,\frac{l^2}{4\nu}\right)^{1/3} \tag{2.18}$$

From (2.18) it is easy to see that $N_{Re} \to \infty$ and $q \to 0$ for $R \to \infty$. The oscillations of the pendulum differ in this case sharply from the oscillations of an analogous pendulum with an ideal liquid. Thus a pendulum filled with water at $t = 20^\circ C$ ($\nu = 1.01 \times 10^{-6}$ m^2/sec) with a radius of the cavity $R = 0.1$ m and a length $l = 0.00016$ m will make five oscillations in a second, while an equal pendulum with an ideal liquid will make 25 oscillations in a second. With the above indicated parameters of the pendulum we have

$$q \approx 0.0012, \qquad N_{\mathrm{Re}} \approx 10^4$$

which fully justifies the use of the solution (2.17) in this case.

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